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COMPUTATIONAL ASPECTS OF THE LOGARITHMIC STRAIN SPACE DESCRIPTION

K. HEIDUSCHKE
Swiss Federal Institute of Technology (ETH), 8092-Zurich, Switzerland

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Abstract—Computational aspects of the logarithmic strain space description are discussed and compared with so-called "updated Lagrangian descriptions". The shortcomings of the latter are demonstrated on an analytical example of homogeneous finite distortion kinematics. The numerical implementation and algorithms of the logarithmic strain space description with respect to the reference configuration are presented. Simulation results of two special purpose finite element programs, which are based on the logarithmic strain space description, are shown.

NOTATION

The following definitions for second order tensors $\mathbf{T}$ are introduced and the Einstein summation convention is adopted:

$\mathbf{T}^{\mathrm{T}}$	transpose of $\mathbf{T}$
$\mathbf{T}^{-1}$	inverse of $\mathbf{T}$
$\mathbf{T}^{-\mathrm{T}} = (\mathbf{T}^{-1})^{\mathrm{T}} = (\mathbf{T}^{\mathrm{T}})^{-1}$	inverse and transpose of $\mathbf{T}$
$\lvert\mathbf{T}\rvert$	determinant of $\mathbf{T}$
T_{ij}	components of $\mathbf{T}$
$T_{kk} = T_{11}+T_{22}+T_{33}$	trace of $\mathbf{T}$
$\mathbf{T}' = \mathbf{T} - 1/3 T_{kk}\mathbf{1}$	deviator of $\mathbf{T}$ (the second order unit tensor is denoted by $\mathbf{1}$)
$\lVert\mathbf{T}\rVert = \sqrt{T_{ij}T_{ij}}$	norm of $\mathbf{T}$

Standard continuum mechanics notation is used, as in Heiduschke (1995):

κ_0 or $\hat{\kappa}$	reference or current configuration of body B
$\mathbf{X}$ or $\mathbf{x}$	position vector of a material point P in κ_0 or $\hat{\kappa}$
t	time
$\mathbf{x}(\mathbf{X}, t)$	motion of body B in a reference description
$\mathbf{F} = \dfrac{\partial \mathbf{x}}{\partial \mathbf{X}}$	deformation gradient, $\lvert\mathbf{F}\rvert > 0$
$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$	symmetric positive definite right Cauchy–Green tensor with respect to κ_0
$\mathbf{F} = \mathbf{R}\mathbf{U} = \hat{\mathbf{V}}\mathbf{R}$	polar decomposition of $\mathbf{F}$
$\mathbf{R}$	orthonormal rotation tensor
$\mathbf{U} = \sqrt{\mathbf{C}}$ or $\hat{\mathbf{V}} = \sqrt{\mathbf{F}\mathbf{F}^{\mathrm{T}}}$	symmetric positive definite right or left stretch tensor with respect to κ_0 or $\hat{\kappa}$
$\boldsymbol{\varepsilon} = \ln(\mathbf{U})$ or $\hat{\boldsymbol{\varepsilon}} = \ln(\hat{\mathbf{V}})$	logarithmic strain tensor with respect to κ_0 or $\hat{\kappa}$
$\dot{\mathbf{x}}$	velocity of a material point P
$\hat{\mathbf{G}} = \dfrac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}}$	velocity gradient with respect to $\hat{\kappa}$
$\hat{\mathbf{D}} = 1/2(\hat{\mathbf{G}} + \hat{\mathbf{G}}^{\mathrm{T}})$	rate of deformation tensor with respect to $\hat{\kappa}$, symmetric part of $\hat{\mathbf{G}}$
$\mathbf{D} = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{D}}\mathbf{R}$	symmetric back-rotated rate of deformation tensor with respect to κ_0
$\dot{\boldsymbol{\varepsilon}}$	logarithmic strain rate with respect to κ_0
$\overset{\Delta}{\boldsymbol{\varepsilon}} = \mathbf{R}\dot{\boldsymbol{\varepsilon}}\mathbf{R}^{\mathrm{T}}$	logarithmic strain rate in $\hat{\kappa}$ co-rotated with respect to the material
$\hat{\mathbf{T}}$	symmetric Cauchy stress with respect to $\hat{\kappa}$
$\mathbf{T} = \mathbf{R}^{\mathrm{T}}\hat{\mathbf{T}}\mathbf{R}$	symmetric back-rotated stress with respect to κ_0
$\boldsymbol{\sigma} = \boldsymbol{\alpha}\mathbf{T}$	symmetric logarithmic stress with respect to κ_0 (work-conjugate to $\boldsymbol{\varepsilon}$)
$\boldsymbol{\alpha}$	kinematical fourth order transformation tensor
$\boldsymbol{\beta}$	inverse to $\boldsymbol{\alpha}$, i.e. $\boldsymbol{\alpha\beta} = \boldsymbol{\beta\alpha} = \mathbf{I}$ (fourth order unit tensor)

Under superposed rigid body rotations the tensors with respect to the current configuration $\hat{\kappa}$, which are marked with a superscript hat, are altered, whereas the tensors with respect to the reference configuration κ_0 are not.

1. INTRODUCTION

The logarithmic strain space description is well-suited to describing finite (elasto-)plasticity, as discussed analytically by Heiduschke (1995). In the work presented we focus on the computational aspects of the logarithmic strain space description and its numerical implementation. By means of an example of homogeneous finite distortion we compare the logarithmic strain space description with so-called "updated Lagrangian descriptions". This example calls into question the general applicability of updated descriptions.

We follow the general thermodynamical framework of Green and Naghdi (1965) and the strain space setting of Naghdi and Trapp (1975), where the yield (or loading) functions are formulated with Green Lagrange strains, which we replace by logarithmic strains. According to Casey and Naghdi (1983), we recall the non-equivalence of stress space and strain space formulations, and according to Naghdi (1990, p. 337), the primacy of the strain space description. We suggest an elasto-plasticity description in the logarithmic (or Hencky) strain space with respect to the reference configuration, which includes elastic and plastic anisotropy. In line with Green and Naghdi (1965) and Naghdi and Trapp (1975), we postulate the existence of a logarithmic plastic strain tensor $\boldsymbol{\varepsilon}^p$ with respect to the reference configuration κ_0. For a stress-free state — in the generalized sense of Casey and Naghdi (1992) — the logarithmic plastic strain tensor $\boldsymbol{\varepsilon}^p$ is identical with the total logarithmic strain tensor $\boldsymbol{\varepsilon}$ with respect to the reference configuration κ_0. The evolution of the plastic strain is described by a rate equation, namely the plastic flow rule. We use the summation convention on repeated indices and define the stress power

$$p = \sigma_{IJ}\dot{\varepsilon}_{IJ}$$

as the inner product of the logarithmic strain rate $\dot{\boldsymbol{\varepsilon}}$ and the logarithmic stress $\boldsymbol{\sigma}$. The stress power may be additively split into an irreversible (plastic) contribution

$$p^p = \sigma_{IJ}\dot{\varepsilon}^p_{IJ},$$

defined by the inner product of the logarithmic plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$ and the logarithmic stress $\boldsymbol{\sigma}$, and the remaining reversible (elastic) contribution

$$\dot{\Psi} = p - p^p = \sigma_{IJ}(\dot{\varepsilon}_{IJ} - \dot{\varepsilon}^p_{IJ}), \tag{1}$$

where $\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ denotes the strain energy function. For convenience and in the light of the latter equation, the abbreviation

$$\boldsymbol{\varepsilon}^e := \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p \tag{2}$$

may be denoted as "elastic strain" $\boldsymbol{\varepsilon}^e$; the tensor $\boldsymbol{\varepsilon}^e$ with respect to the reference configuration κ_0 is not defined via any multiplicative elasto-plastic decomposition of the deformation gradient, it is simply the abbreviation (2), namely the difference of the finite logarithmic strains $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^p$. Furthermore, for metals $\boldsymbol{\varepsilon}^e$ is a back-rotated infinitesimal strain, as the corresponding logarithmic stress components

$$\sigma_{IJ} = \frac{\partial \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)}{\partial \varepsilon_{IJ}} \tag{3}$$

are orders of magnitude smaller than the modulus of elasticity.

It should be emphasized that, in contrast to the large body of finite element literature, we do not use the following:

(1) The plastic strain definition of Simo (1988a,b) and Eterovic and Bathe (1990), where the elastic strain tensor is defined as logarithmic strain and where then the abbreviation

$$\boldsymbol{\varepsilon}^p = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^e$$

is introduced — in contrast to eqn (2), where ε^e is defined by $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^p$. Since for metals the "elastic strain" is an "infinitesimal strain", all generalized finite strain tensors with respect to the reference configuration κ_0 [see Doyle and Ericksen (1956); Hill (1968)], including the logarithmic strain, are equal up to the first order. They are back-rotated "infinitesimal strain" tensors so that the definition of logarithmic "elastic strain" of Simo (1988a,b) and Eterovic and Bathe (1990) loses its significance.

(2) So-called "updated Lagrangian descriptions", where the rate of deformation tensor

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p \tag{4}$$

or associated quantities are split into elastic and plastic contributions denoted by the superscripts e and p, respectively. The components of the plastic contributions are integrated (somehow co-rotationally), and with the elastic contributions incremental stress–strain relations are formulated [see, e.g. Goudreau and Hallquist (1982); Nagtegaal (1982); Hughes (1984)]. The questions of appropriate tensor derivatives have been discussed extensively, even though the tensor derivative co-rotated with respect to the material is well-defined by Doyle and Ericksen (1956).

(3) A non-unique multiplicative elasto-plastic decomposition of the deformation gradient

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p, \tag{5}$$

in contrast to Nemat-Nasser (1982), Ortiz (1987), Simo (1988a,b), Eterovic and Bathe (1990), Weber and Anand (1990) and many other publications on numerical methods. The multiplicative decomposition [eqn (5)] was introduced by Kröner (1960) and discussed further by Backman (1964) and Lee and Liu (1967). The non-uniqueness of the decomposition (5) as has been pointed out by Lee (1969) and others.

(4) Stress space descriptions of elasto-plasticity, where stresses are used as independent variables in order to express yield functions, flow and hardening rules.

(5) Elasto-plastic constitutive models formulated without yield (or loading) conditions, as introduced by Bodner (1968) in a multi-dimensional formulation. Questions about the numerical treatment of these models are covered by Ortiz (1987), Weber and Anand (1990), and the references cited therein.

2. COMPARISON OF DIFFERENT DESCRIPTIONS USED FOR FINITE ELEMENTS

In order to compare different descriptions used in the finite element method we study an analytical example of homogeneous finite distortion kinematics — cf. the third example of Heiduschke (1995) — where the current coordinates

$$x = \left(X\cos\left(\frac{\pi t}{2T}\right) - Y\sin\left(\frac{\pi t}{2T}\right)\right)\exp\left(\frac{t}{T}\right)$$

$$y = \left(X\sin\left(\frac{\pi t}{2T}\right) + Y\cos\left(\frac{\pi t}{2T}\right)\right)\exp\left(-\frac{t}{T}\right) \tag{6}$$

are functions of the reference coordinates X, Y and time t. In Fig. 1 the deformation

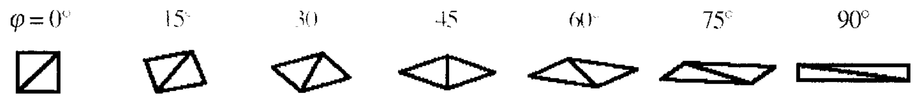

Fig. 1. Homogeneous finite distortion kinematics, where the associated generalized finite strains with respect to the reference configuration have changing principal directions.

process (6) of a unit square in the reference configuration (at $t = 0$) is shown at times or corresponding angles of rigid body rotation

$$\frac{t}{T} = 0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{6}, 1 \quad \text{or} \quad \varphi = 0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ,$$

respectively. The generalized finite strains with respect to κ_0, which result from eqn (6), have changing principal directions so that the logarithmic strain rate and the back-rotated rate of deformation tensors with respect to κ_0 are not identical

$$\mathbf{D} = \boldsymbol{\alpha}\dot{\boldsymbol{\varepsilon}} \quad \text{and} \quad \dot{\boldsymbol{\varepsilon}} = \boldsymbol{\beta}\mathbf{D} \tag{7}$$

or, equivalently, the kinematical fourth order transformation tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, are in general not fourth order unit tensors; see Heiduschke (1995).

We consider the time steps

$${}^{n}t = \frac{n}{N}T,$$

where n, N and T denote the number of the current increment, the total number of increments and the length of the considered time interval, respectively, and denote current time steps by upper left indices. The components of ${}^{n}\mathbf{F}$, ${}^{n}\mathbf{R}$, ${}^{n}\hat{\mathbf{V}}$, ${}^{n}\mathbf{U}$, ${}^{n}\hat{\boldsymbol{\varepsilon}}$, ${}^{n}\boldsymbol{\varepsilon}$, ${}^{n}\hat{\mathbf{D}}$ and ${}^{n}\mathbf{D}$ are then

$${}^{n}F_{iI} = \begin{bmatrix} \exp\left(\frac{n}{N}\right)\cos\left(\frac{\pi n}{2N}\right) & -\exp\left(\frac{n}{N}\right)\sin\left(\frac{\pi n}{2N}\right) \\ \exp\left(-\frac{n}{N}\right)\sin\left(\frac{\pi n}{2N}\right) & \exp\left(-\frac{n}{N}\right)\cos\left(\frac{\pi n}{2N}\right) \end{bmatrix} \tag{8}$$

$${}^{n}R_{iI} = \begin{bmatrix} \cos\left(\frac{\pi n}{2N}\right) & -\sin\left(\frac{\pi n}{2N}\right) \\ \sin\left(\frac{\pi n}{2N}\right) & \cos\left(\frac{\pi n}{2N}\right) \end{bmatrix},$$

$${}^{n}\hat{V}_{ij} = \begin{bmatrix} \exp\left(\frac{n}{N}\right) & 0 \\ 0 & \exp\left(-\frac{n}{N}\right) \end{bmatrix}, \quad {}^{n}U_{IJ} = {}^{n}R_{iI}\,{}^{n}\hat{V}_{ij}\,{}^{n}R_{jJ},$$

$${}^{n}\hat{\varepsilon}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{n}{N}, \quad {}^{n}\varepsilon_{IJ} = {}^{n}R_{iI}\,{}^{n}\hat{\varepsilon}_{ij}\,{}^{n}R_{jJ}, \tag{9}$$

$${}^{n}\hat{D}_{ij} = \begin{bmatrix} \frac{1}{T} & -\frac{\pi}{2T}\sinh\left(\frac{2n}{N}\right) \\ -\frac{\pi}{2T}\sinh\left(\frac{2n}{N}\right) & -\frac{1}{T} \end{bmatrix}, \quad {}^{n}D_{IJ} = {}^{n}R_{iI}\,{}^{n}\hat{D}_{ij}\,{}^{n}R_{jJ}$$

and the components of the time derivatives of the logarithmic strain tensors co-rotated with respect to the material are

$$ {}^n\hat{\dot{\boldsymbol{\varepsilon}}}_{ij} = \begin{bmatrix} \dfrac{1}{T} & -\dfrac{\pi n}{TN} \\ -\dfrac{\pi n}{TN} & -\dfrac{1}{T} \end{bmatrix}, \quad {}^n\dot{\varepsilon}_{IJ} = {}^nR_{iI}\,{}^n\hat{\dot{\varepsilon}}_{ij}\,{}^nR_{jJ}. $$

We define the following integrals, approximated by the sums of the midpoint rates, which are indicated by the index shift $n+0.5$: the integral of the logarithmic strain rate with respect to κ_0

$$ \mathbf{I}^{\varepsilon} = \int_0^t \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}t \cong \sum_{n=0}^{N-1} {}^{n+0.5}\dot{\boldsymbol{\varepsilon}}\,\frac{T}{N}; \tag{10} $$

the integral co-rotated with respect to the material of the logarithmic strain rate with respect to $\hat{\kappa}$

$$ \mathbf{I}^{\hat{\varepsilon}} = \mathbf{R}\mathbf{I}^{\varepsilon}\mathbf{R}^{\mathrm{T}}; \tag{11} $$

the integral of the components of the back-rotated rate of deformation tensor with respect to κ_0

$$ \mathbf{I}^{\mathrm{D}} = \int_0^t \mathbf{D}\,\mathrm{d}t \cong \sum_{n=0}^{N-1} {}^{n+0.5}\mathbf{D}\,\frac{T}{N}; \tag{12} $$

and the integral of the components of the rate of deformation tensor with respect to $\hat{\kappa}$

$$ \int_0^t \hat{\mathbf{D}}\,\mathrm{d}t \cong \sum_{n=0}^{N-1} {}^{n+0.5}\hat{\mathbf{D}}\,\frac{T}{N}. \tag{13} $$

For a material description, the latter integral (13) makes no sense, since the components $\hat{D}_{ij}$, which are altered under superposed rigid body motions, are only time-integrated. Hence, the integral (13) depends on superposed rigid body rotations and, in general, it is not identical to $\mathbf{R}\,\mathbf{I}^{\mathrm{D}}\,\mathbf{R}^{\mathrm{T}}$, the corresponding integral co-rotated with respect to the material. The integral (13) is introduced for reasons of a negative comparison. Furthermore, the following updated strain increments—as they appear in the so-called "updated Lagrangian description" [see, e.g. Goudreau and Hallquist (1982); Nagtegaal (1982); Hughes (1984)]—are expressed by the deformation gradients of the current ${}^{n+1}\mathbf{F}$ and previous time steps ${}^n\mathbf{F}$ [eqn (8)]: the small-strain increment†

$$ {}^n\Delta\mathbf{E}^{\mathrm{S}} = \tfrac{1}{2}({}^{n+1}\mathbf{F}\,{}^n\mathbf{F}^{-1} + {}^n\mathbf{F}^{-\mathrm{T}}\,{}^{n+1}\mathbf{F}^{\mathrm{T}}) - \mathbf{1}, $$

the midpoint-strain increment—as defined for $\alpha = 1/2$ in eqns (93)–(96) of Hughes (1984)—

$$ {}^n\Delta\mathbf{E}^{\mathrm{M}} = ({}^{n+1}\mathbf{F}\,{}^n\mathbf{F}^{-1} - \mathbf{1})({}^{n+1}\mathbf{F}\,{}^n\mathbf{F}^{-1} + \mathbf{1})^{-1} + ({}^{n+1}\mathbf{F}\,{}^n\mathbf{F}^{-1} + \mathbf{1})^{-\mathrm{T}}({}^n\mathbf{F}^{-\mathrm{T}}\,{}^{n+1}\mathbf{F}^{\mathrm{T}} - \mathbf{1}) $$

and the Green–Lagrange strain increment—as defined in eqn (9) of Nagtegaal (1982)—

$$ {}^n\Delta\mathbf{E}^{\mathrm{G}} = \tfrac{1}{2}({}^n\mathbf{F}^{-\mathrm{T}}\,{}^{n+1}\mathbf{F}^{\mathrm{T}}\,{}^{n+1}\mathbf{F}\,{}^n\mathbf{F}^{-1} - \mathbf{1}). $$

†The Cartesian components of the small-strain increments—if expressed by the displacements u_i of the updated configurations—are $\Delta E^{\mathrm{S}}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$.

The sums which are associated with the updated strain increments $\Delta\mathbf{E}^{S}$, $\Delta\mathbf{E}^{M}$, $\Delta\mathbf{E}^{G}$ are defined by

$$\boldsymbol{\Sigma}^{S} = \sum_{n=0}^{N-1} {}^{n}\Delta\mathbf{E}^{S}, \quad \boldsymbol{\Sigma}^{M} = \sum_{n=0}^{N-1} {}^{n}\Delta\mathbf{E}^{M}, \quad \boldsymbol{\Sigma}^{G} = \sum_{n=0}^{N-1} {}^{n}\Delta\mathbf{E}^{G}, \tag{14}$$

respectively. We consider the deformation process depicted in Fig. 1 for a time interval of $T = 1$ s and a discretization of $N = 90$ time steps so that the rigid body rotation is 1° per increment. We compare the xx, xy and yy components and the norm of the quantities (9)–(14) as depicted in Figs 2–5.

Equations (9)–(14) can be divided into four groups:

G1 : $\boldsymbol{\varepsilon}$, $\mathbf{I}^{\varepsilon}$. The logarithmic strain tensor with respect to κ_0 is the basic reference of the comparison presented. The integral $\mathbf{I}^{\varepsilon}$ of its rate is identical to the tensor $\boldsymbol{\varepsilon}$.

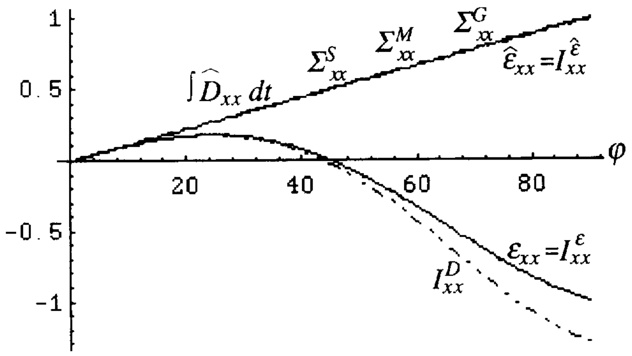

Fig. 2. Comparison of the xx components.

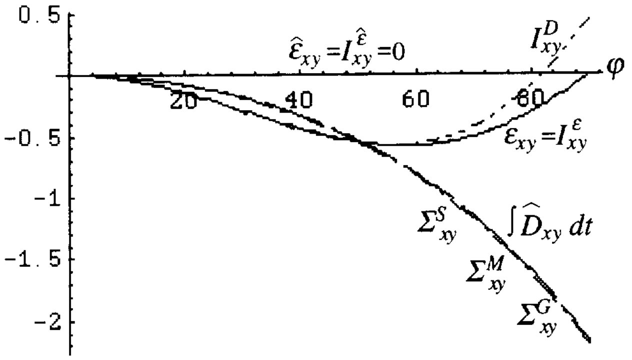

Fig. 3. Comparison of the xy components.

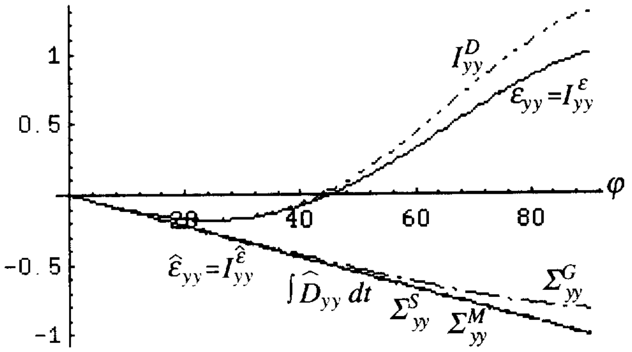

Fig. 4. Comparison of the yy components.

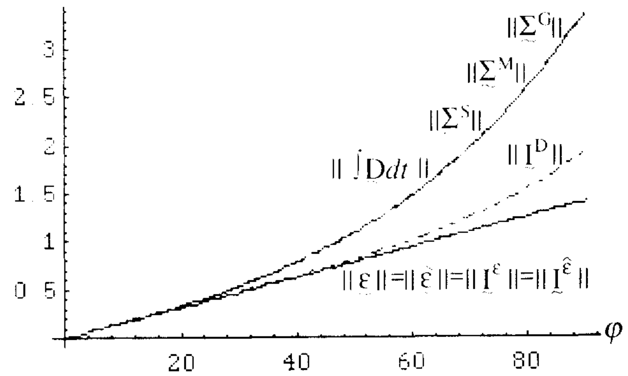

Fig. 5. Comparison of the norms [illegible]

G2: $\hat{\varepsilon}$, $\mathbf{I}^{\hat{\varepsilon}}$. The logarithmic strain tensor with respect to $\hat{\kappa}$ is introduced as reference for tensors with respect to $\hat{\kappa}$. It should be noted that the definition of its rate and integral are based on ε and $\mathbf{I}^{\varepsilon}$, respectively. Therefore rate and integral are termed "co-rotated with respect to the material". The integral $\mathbf{I}^{\hat{\varepsilon}}$ is identical to the tensor $\hat{\varepsilon}$.

G3: $\mathbf{I}^D$. The time-integral of the back-rotated rate of deformation tensor with respect to κ_0 reveals characteristics similar to those of ε. However, due to eqn (7), its components and norm deviate from the exact solution. The integral $\mathbf{I}^D$ depends not only on the current and reference configurations, but also on the path.†

G4: $\mathbf{\Sigma}^S$, $\mathbf{\Sigma}^M$, $\mathbf{\Sigma}^G$, $\int\hat{\mathbf{D}}dt$. The sums of the strain increments (14) of so-called "updated Lagrangian descriptions" graphically behave‡ like $\int\hat{\mathbf{D}}dt$, the component-wise time-integral of $\hat{\mathbf{D}}$. In particular, their shear component (see Fig. 3) and their norm (see Fig. 5) totally differ from $\hat{\varepsilon}$, which they are supposed to approximate. The sums $\mathbf{\Sigma}^S$, $\mathbf{\Sigma}^M$, $\mathbf{\Sigma}^G$ depend not only on the current and reference configurations: they are path-dependent quantities which cannot be approximations of generalized finite strain tensors (with respect to the current configuration $\hat{\kappa}$).

The kinematic example presented above can be interpreted with respect to the logarithmic plastic strain tensor ε^p. Plastic strain should depend only on the current and reference stress-free configurations, in the generalized sense of Casey and Naghdi (1992). In rate-type theories the plastic strain ε^p is time-integrated from its rate, which is given by the plastic flow rule. For rigid plastic material, where total and plastic (logarithmic) strain tensors are identical, $\varepsilon = \varepsilon^p$, the example presented above can be used to compare integrals sums of different plastic flow increments, namely $\dot{\varepsilon}\Delta t$, $\mathbf{D}\Delta t$, $\hat{\mathbf{D}}\Delta t$, $\Delta\mathbf{E}_S$, $\Delta\mathbf{E}_M$, $\Delta\mathbf{E}_G$. It turns out that from the corresponding integrals sums only $\mathbf{I}^{\varepsilon}$ or its rotation to the current configuration $\mathbf{I}^{\hat{\varepsilon}}$ result in ε or $\hat{\varepsilon}$, respectively. The other integrals sums $\mathbf{I}^D$, $\mathbf{\Sigma}^S$, $\mathbf{\Sigma}^M$, $\mathbf{\Sigma}^G$, $\int\hat{\mathbf{D}}dt$ which are used in so-called "updated Lagrangian descriptions", result in neither ε nor $\hat{\varepsilon}$. They result in so-called internal variables, which cannot be regarded as plastic strain tensors in the sense of ε^p or $\hat{\varepsilon}^p = \mathbf{R}\varepsilon^p\mathbf{R}^T$ (see Figs 3 and 5).

LOGARITHMIC STRAIN SPACE ALGORITHMS

A constitutive algorithm based on the logarithmic strain space will not have the shortcomings of updated descriptions, which have been discussed in the previous example. Furthermore, logarithmic strain tensors have the following features [see Heiduschke (1995)]: traces and deviators of logarithmic strain tensors describe finite dilatation and finite distortion, respectively. Hence, by using logarithmic strains the finite dilatation and distortion can be additively decoupled, plastic incompressibility can be introduced by

†The deviations depicted in Figs 2–5 are not of a numerical nature, i.e. do not primarily derive from the discretization N. The deviations are of physical nature and appear even for $N \to \infty$.

‡The sum $\mathbf{\Sigma}^G$ in Figs 2 and 4 shows in a small way the influence of the non-symmetric tension compression characteristics of the incremental Green–Lagrange strain. This influence is of a numerical nature and disappears for finer time-discretizations.

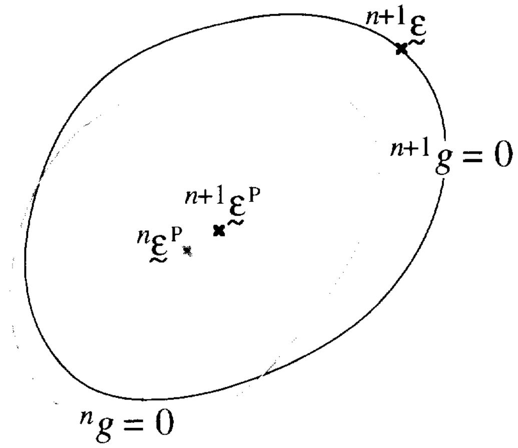

Fig. 6. Yield surface in the strain space at the previous and current time steps.

enforcing the plastic logarithmic strain $\boldsymbol{\varepsilon}^{P}$ to be deviatoric and a von Mises type of yield function† can be expressed by the second invariant of $(\boldsymbol{\varepsilon}'-\boldsymbol{\varepsilon}^{P})$, in analogy to the infinitesimal deformation theory.

In general, the time-discrete constitutive algorithm follows from time-continuous constitutive equations, which are described in the logarithmic strain space. In a logarithmic strain space formulation the yield (or loading) function $g(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{P}, k)$ depends only on the symmetric total logarithmic strain $\boldsymbol{\varepsilon}$, the symmetric plastic logarithmic strain $\boldsymbol{\varepsilon}^{P}$ and the hardening parameter(s) k. The yield surface $g = 0$ is a five-dimensional hypersurface bounding a six-dimensional convex set in strain space as depicted in Fig. 6. For the discretization we consider a yield function g which is monotonic in a suitable sense‡—which also includes a Tresca type of yield function. The constitutive algorithm is entered with the current total strain $^{n+1}\boldsymbol{\varepsilon}$, where the upper left index denotes the current time step. In the time-discrete six-dimensional strain space the current total strain $^{n+1}\boldsymbol{\varepsilon}$ may lie inside the yield surface ^{n}g of the previous time step, $g^{\text{trial}} < 0$, on it, $g^{\text{trial}} = 0$, or outside it, $g^{\text{trial}} > 0$, where the time-discrete trial function is defined as $g^{\text{trial}} = g(^{n+1}\boldsymbol{\varepsilon}, {}^{n}\boldsymbol{\varepsilon}^{P}, {}^{n}k)$. In the first two cases, $g^{\text{trial}} \leqslant 0$, which correspond to elasticity (or unloading) and neutral loading, no plastic flow occurs; in the latter case, $g^{\text{trial}} > 0$, plastic flow occurs. In the time-discrete strain space the integration of the elasto-plastic constitutive equations reduces to the (hyper)geometrical problem of finding the current yield surface $^{n+1}g = 0$, which goes through $^{n+1}\boldsymbol{\varepsilon}$ such that the plastic increment

$$\Delta\boldsymbol{\varepsilon}^{P} = {}^{n+1}\boldsymbol{\varepsilon}^{P} - {}^{n}\boldsymbol{\varepsilon}^{P}$$

obeys the flow rule, the hardening law and the consistency condition, $\dot{g} = 0$. An illustration of an isotropic elasto-plasticity algorithm is presented in the Appendix.

It should be noted that no stress tensor is used for the integration of the constitutive equations. If the stress is desired, it can be calculated at the end of the algorithm, i.e. after the time integration of the constitutive equations. The logarithmic stress $\boldsymbol{\sigma}$ follows from eqn (3). It may be transformed to Cauchy stress $\hat{\mathbf{T}}$ using

$$\hat{\mathbf{T}} = \mathbf{RTR}^{T} \quad \text{and} \quad \mathbf{T} = \beta\boldsymbol{\sigma}. \tag{15}$$

4. IMPLEMENTATION INTO FINITE ELEMENT MODELS

In isoparametric kinematic finite elements current and reference coordinates

$$x_i = \bar{x}_{iN}H_N \quad \text{and} \quad X_J = \bar{X}_{JN}H_N \tag{16}$$

†In a von Mises type of yield function plastic flow occurs at a certain elastic distortional energy.
‡Note, for instance, that the yield function g has a minimum as its only critical point.

are interpolated with the same shape functions $H_N = H_N(l_i)$, respectively, which depend only on the local variables l_i. From the approximations (16) the components of the deformation gradient

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} = \frac{\partial x_i}{\partial l_k}\left[\frac{\partial X_J}{\partial l_k}\right]^{-1} = \bar{x}_{iN}\frac{\partial H_N}{\partial l_k}\left[\bar{X}_{JM}\frac{\partial H_M}{\partial l_k}\right]^{-1} \tag{17}$$

and the velocity gradient

$$G_{ij} = \frac{\partial \dot{x}_i}{\partial x_j} = \frac{\partial \dot{x}_i}{\partial l_k}\left[\frac{\partial x_j}{\partial l_k}\right]^{-1} = \dot{\bar{x}}_{iN}\frac{\partial H_N}{\partial l_k}\left[\bar{x}_{jM}\frac{\partial H_M}{\partial l_k}\right]^{-1} \tag{18}$$

can be calculated. The approximated deformation gradient (17) and velocity gradient (18) are used in the following finite element algorithm, which must be applied in each time step for each integration point of the finite element model:

(1) Calculate the components ${}^{n+1}R_{iJ}$ of the rotation tensor at the current time step ${}^{n+1}$ by using the approximated deformation gradient (17).

(2) Calculate the components ${}^{n+1}\varepsilon_{IJ}$ of the total logarithmic strain tensor $\boldsymbol{\varepsilon} = 1/2\ \ln(\mathbf{C})$ at the current time step ${}^{n+1}$ by using a transformation to the principal axes and back again, $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\ \mathbf{F}$ and the approximated deformation gradient (17). Furthermore, determine the coefficients of ${}^{n+1}\boldsymbol{\beta}$ as described by Heiduschke (1995).

(3) From the constitutive algorithm, which is formulated in the logarithmic strain space and of which an example is presented in the Appendix (see Fig. A2), find the current plastic state, i.e. the components ${}^{n+1}\varepsilon^{\mathrm{p}}_{IJ}$ of the plastic strain tensor and the hardening parameter ${}^{n+1}k$.

(4) Calculate the components ${}^{n+1}\boldsymbol{\sigma}$ of the logarithmic stress by using the stress–strain relations (3). By use of eqn (15) the logarithmic stress is transformed to the Cauchy stress ${}^{n+1}\hat{\mathbf{T}}$.

(5) Apply the principle of virtual work in the current configuration—using ${}^{n+1}\hat{\mathbf{T}}$ and ${}^{n+1}\mathbf{G}$ [eqn (18)]—in order to derive the appropriate finite element matrices for the calculation of the equivalent nodal force vectors.

(6) If desired, the tangential element stiffness is assembled over the element's integration points by using the tangential elasto-plastic matrix—the symmetric derivative of eqn (3) with the plastic flow $\dot{\varepsilon}^{\mathrm{p}}$ inserted—and transforming it, via eqns (7) and (15), to the current configuration κ where the principle of virtual work is applied.

5. APPLICATIONS

The logarithmic strain space description has successfully been applied in the finite element programs AutoForm and Pafix, which have been presented by Heiduschke *et al.* (1991) and Anderheggen *et al.* (1994):

AutoForm is a special-purpose program for simulating the sheet metal forming process. It is based on an implicit formulation of the static nodal equilibrium and requires the solution of a global system matrix. Due to the two-dimensional shape of sheet metal, a decoupling into stretching and bending is performed for the system solution, which is then much better conditioned than without the decoupling. The sheet metal is modelled by plane triangular elements with three nodes, which may be stacked in layers. In critical areas the element mesh is automatically refined by recursively subdividing one triangle into four similar triangles. Typical simulation results of AutoForm are given in Figs 7 and 8, where the initial and final shape with the finite element mesh of a car door are shown. The two door handle recesses are modelled in order to study two variants in one simulation. The magnitudes of the strains encountered are moderate.

Pafix is a special purpose program for simulating the penetration of a metal nail into a metal substratum. It is based on a dynamic nodal equilibrium which takes into account inertia effects and, hence, wave propagation. It is implemented using an explicit time

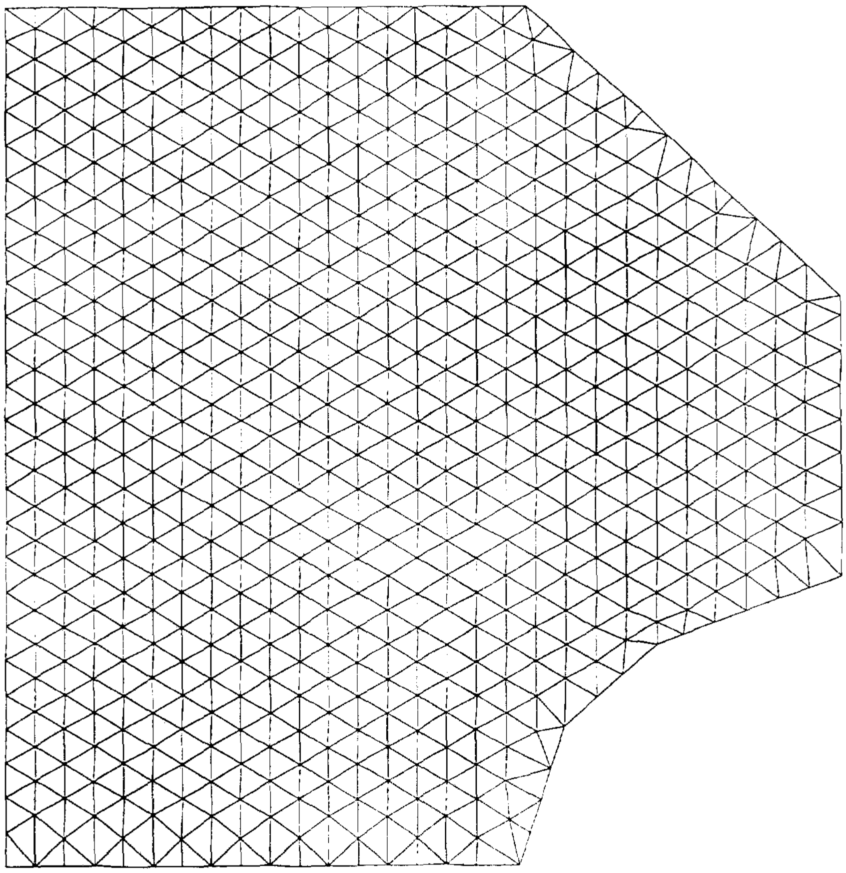

Fig. 7. Initial shape and finite element mesh of a metal sheet.

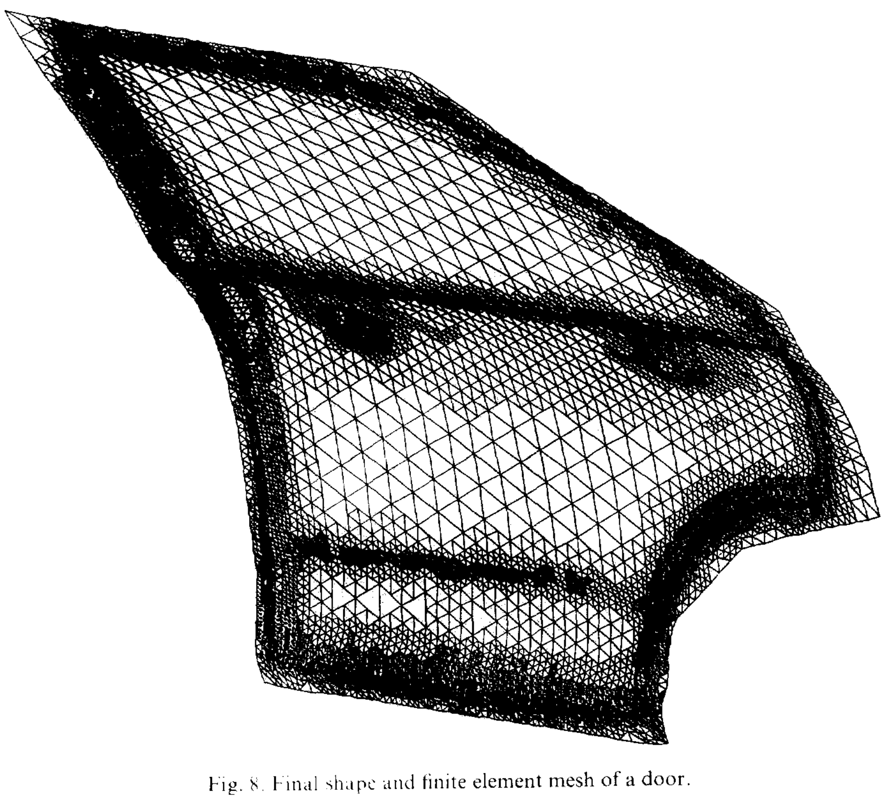

Fig. 8. Final shape and finite element mesh of a door.

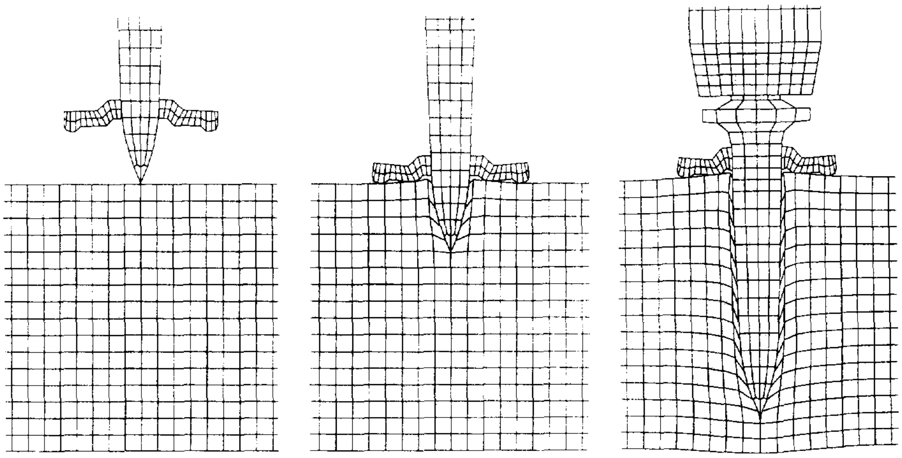

Fig. 9. Metal nail with tophat penetrating a metal substratum.

integration scheme so that a solution of the global system matrix is not required. Furthermore, rotational symmetry of the nail, the substratum and all other components, e.g. tophats, is assumed. Typical simulation results of Pafix are depicted in Fig. 9, where the initial configuration of the nail, tophat and substratum and the penetrated nail are shown. The magnitudes of the strains encountered are large, especially nearby the axis of rotational symmetry.

6. CONCLUSIONS

The presented description of an elasto-plastic continuum undergoing finite deformation and its implementation into finite element codes is based on the logarithmic strain space with respect to the reference configuration κ_0. The material is therefore described with respect to its reference configuration κ_0—it is a so-called "total Lagrangian material description". The time integration of the flow rule and the hardening law is performed directly in the logarithmic strain space and it is based on the total strain, not its increment. The plastic deformation is also described by the logarithmic strain ε^p, which is a physically well-defined measure.

In contrast to the above, the so-called "updated Lagrangian description" is an incremental formulation which is based on the current configuration $\hat{\kappa}$ and which uses co-rotated rates of second order tensors. In particular the (somehow co-rotationally) integrated components of plastic strain rates cannot be the components of the logarithmic plastic strain tensors ε^p or $\hat{\varepsilon}^p$, as shown by the comparison presented in Section 2 (see Figs 3 and 5). If the so-called midpoint strain increment of Hughes (1984) is used, which is defined with respect to the middle of the time (or load) step, then the strain is defined in space and by its (time) increment; but strains are generally defined only with respect to space and should be independent of the (time) increment!

Furthermore, in the so-called "updated Lagrangian description", deformation-induced anisotropy and material anisotropy are mixed. Consider, for example, a laminate which undergoes the homogeneous finite deformation depicted in Fig. 10. If the material is

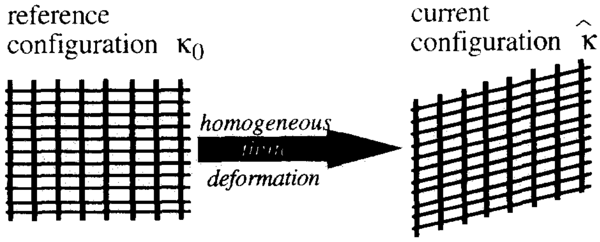

Fig. 10. Homogeneous finite deformation of a laminate.

described with respect to the reference configuration κ_0 its behaviour is orthotropic; if the material is described with respect to the current configuration $\hat{\kappa}$ its behaviour is no longer orthotropic, but fully anisotropic. In $\hat{\kappa}$ there are no longer any orthogonal principal material directions and hence the influences of deformation and material behaviour on the anisotropy are mixed.

The description presented here does not exhibit the problems of the updated descriptions, namely the mixture of deformation-induced and material anisotropy and the rates and their respective integration, because the description is based on the reference configuration κ_0 and it uses the total strain in the time-discrete strain space and not the strain increment. Furthermore, the use of the logarithmic strains enables an additive decoupling of finite dilatation and distortion, even though the evaluation of the components of $\boldsymbol{\varepsilon}$, (and $\boldsymbol{\beta}$) involves a transformation to principal axes and back again, which numerically requires some effort. As outlined in the present work we describe the plasticity and integrate the plastic rate in the logarithmic strain space, and we therefore do not need any stresses, which are dependent variables in a strain space setting.

Finally, even though we do not use the additive split (4) of the rate of deformation tensor $\hat{\mathbf{D}}$ or of the according the back-rotated rate $\mathbf{D}$ in our considerations, eqn (4) may be based on the additive split of the stress power (1) [see also Nemat-Nasser (1982)] by defining the plastic and elastic parts of $\hat{\mathbf{D}}$ and $\mathbf{D}$ via

$$\hat{\mathbf{D}}^{p} = \mathbf{R}\mathbf{D}^{p}\mathbf{R}^{T} \quad \text{and} \quad \mathbf{D}^{p} = \boldsymbol{\alpha}\dot{\boldsymbol{\varepsilon}}^{p},$$

$$\hat{\mathbf{D}}^{e} = \mathbf{R}\mathbf{D}^{e}\mathbf{R}^{T} \quad \text{and} \quad \mathbf{D}^{e} = \boldsymbol{\alpha}(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^{p}),$$

respectively, in contrast to Lee (1981), where the elastic and plastic parts of the rate of deformation tensors are defined via the not unique multiplicative elasto-plastic decomposition of the deformation gradient (5), which is not additively decoupled—as indicated by eqn (2.17) of Lee (1981).

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APPENDIX

An illustration of a constitutive algorithm based on a strain space description is presented for isotropic elastoplasticity with a yield (or loading) function of von Mises type, an associated flow rule, isotropic hardening and isotropic elasticity [see Heiduschke (1989)]. The hardening behaviour is specified by the piecewise linear hardening function of Fig. A1, where $\bar{\sigma}$ denotes the equivalent stress and $\bar{\varepsilon}^p$ the equivalent plastic strain. For isotropic elasto–plasticity we use the equivalent plastic strain $\bar{\varepsilon}^p$ as the independent hardening parameter instead of k, which has been used in Section 3 for the general description. The structogram of the plasticity algorithm is depicted in Fig. A2. In the algorithm we use deviators of the logarithmic strain tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^p$. The traces and deviators of the logarithmic strain tensors are measures of finite dilatation and finite distortion, respectively. They are independent in the sense $\varepsilon'_{ij}\varepsilon''_{ij} = 0$, when denoting the spherical part of a tensor $\boldsymbol{\varepsilon}$ by

$$\boldsymbol{\varepsilon}'' = \tfrac{1}{3}\varepsilon_{KK}\mathbf{1}.$$

According to the flow rule the plastic strain increment is deviatoric, as it is proportional to a deviator, and so is its time integral, the plastic strain $\boldsymbol{\varepsilon}^p = \boldsymbol{\varepsilon}'^p$. Hence, the incompressibility condition of finite plastic deformation $\varepsilon^p_{KK} = 0$ is fulfilled. In the structogram of Fig. A2 we use the tensor norm, defined in the Notation, and the shear modulus

$$G = \frac{1}{2}\frac{E}{1+\nu},$$

where Young's modulus is denoted by E and Poisson's ratio by ν. The if-block describes the yield condition, $\omega \geqslant 1$ corresponds to elasticity (including unloading and neutral loading) and $\omega < 1$ to plasticity. If plasticity occurs, the steps of the blocks which are marked by an (R) in the lower right corner must be repeated and properly corrected until the equivalent plastic strain ${}^{n+1}\bar{\varepsilon}^p$ is in the current hardening range. From the hardening parameter

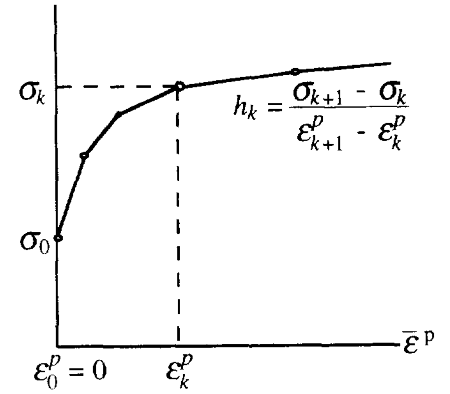

Fig. A1. The piecewise linear hardening function $\bar{\sigma}(\bar{\varepsilon}^p)$ and the hardening slope h_k from a tensile test.

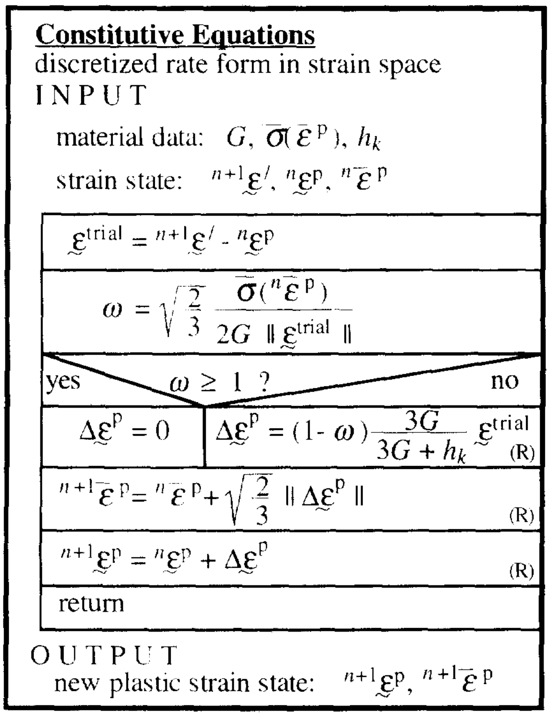

Fig. A2. Structogram of a strain space algorithm for isotropic elasto-plastic constitutive equations formulated in deviators.

$\bar{\varepsilon}^p$ and by use of Fig. A1 the equivalent flow stress $\bar{\sigma}$ can always be obtained. The equivalent flow stress $\bar{\sigma}$ is basically a measure for the radius of the von Mises cylinder in principal strain space.

For isotropic elasticity the general stress strain relations (3) may further be specified by

$$\sigma = \frac{E}{1+\nu}(\varepsilon - \varepsilon^p) \quad \text{and} \quad \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}.$$